

# 1 Diagonal and Off-Diagonal Coefficients

The partition function  $\mathcal{Z}$  is given by

$$\mathcal{Z} = \int \mathcal{D}U [\det M(\mu_u)]^{1/4} [\det M(\mu_d)]^{1/4} [\det M(\mu_s)]^{1/4} \exp\{-S_G\} . \quad (1)$$

We denote the derivatives of  $\mathcal{Z}$  with respect to the quark chemical potentials by

$$A_i^f = \frac{1}{\mathcal{Z}} \frac{\partial^i}{\partial \mu_f^i} \mathcal{Z} , \quad A_{ij}^{fg} = \frac{1}{\mathcal{Z}} \frac{\partial^i}{\partial \mu_f^i} \frac{\partial^j}{\partial \mu_g^j} \mathcal{Z} , \quad A_{ijk}^{fgh} = \frac{1}{\mathcal{Z}} \frac{\partial^i}{\partial \mu_f^i} \frac{\partial^j}{\partial \mu_g^j} \frac{\partial^k}{\partial \mu_h^k} \mathcal{Z} , \quad (2)$$

and the derivatives of  $\ln \mathcal{Z}$  by

$$(\ln \mathcal{Z})_i^f = \frac{\partial^i}{\partial \mu_f^i} \ln \mathcal{Z} , \quad (\ln \mathcal{Z})_{ij}^{fg} = \frac{\partial^i}{\partial \mu_f^i} \frac{\partial^j}{\partial \mu_g^j} \ln \mathcal{Z} , \quad (\ln \mathcal{Z})_{ijk}^{fgh} = \frac{\partial^i}{\partial \mu_f^i} \frac{\partial^j}{\partial \mu_g^j} \frac{\partial^k}{\partial \mu_h^k} \ln \mathcal{Z} , \quad (3)$$

where  $f, g, h \in \{u, d, s\}$ . Derivatives are taken at  $\mu_u = \mu_d = \mu_s = 0$ . Due to the particle anti-particle symmetry we have  $A_i^f = 0$  with  $i$  odd,  $A_{ij}^{fg} = 0$  with  $i + j$  odd, and  $A_{ijk}^{fgh} = 0$  with  $i + j + k$  odd. We find

$$(\ln \mathcal{Z})_2^f = A_2^f \quad (4)$$

$$(\ln \mathcal{Z})_4^f = -3A_2^{f^2} + A_4^f \quad (5)$$

$$(\ln \mathcal{Z})_6^f = 30A_2^{f^3} - 15A_2^f A_4^f + A_6^f \quad (6)$$

$$(\ln \mathcal{Z})_8^f = -630A_2^{f^4} + 420A_2^{f^2} A_4^f - 35A_4^{f^2} - 28A_2^f A_6^f + A_8^f \quad (7)$$

$$(\ln \mathcal{Z})_{11}^{fg} = A_{11}^{fg} \quad (8)$$

$$(\ln \mathcal{Z})_{31}^{fg} = -3A_{11}^{fg} A_{20}^{fg} + A_{31}^{fg} \quad (9)$$

$$(\ln \mathcal{Z})_{51}^{fg} = 30A_{11}^{fg} A_{20}^{fg^2} - 10A_{20}^{fg} A_{31}^{fg} - 5A_{11}^{fg} A_{40}^{fg} + A_{51}^{fg} \quad (10)$$

$$(\ln \mathcal{Z})_{71}^{fg} = -630A_{11}^{fg} A_{20}^{fg^3} + 210A_{20}^{fg^2} A_{31}^{fg} + 210A_{11}^{fg} A_{20}^{fg} A_{40}^{fg} - 35A_{31}^{fg} A_{40}^{fg} - 21A_{20}^{fg} A_{51}^{fg} - 7A_{11}^{fg} A_{60}^{fg} + A_{71}^{fg} \quad (11)$$

$$(\ln \mathcal{Z})_{22}^{fg} = -2A_{11}^{fg^2} - A_{02}^{fg} A_{20}^{fg} + A_{22}^{fg} \quad (12)$$

$$(\ln \mathcal{Z})_{42}^{fg} = 24A_{11}^{fg^2} A_{20}^{fg} + 6A_{02}^{fg} A_{20}^{fg^2} - 6A_{20}^{fg} A_{22}^{fg} - 8A_{11}^{fg} A_{31}^{fg} - A_{02}^{fg} A_{40}^{fg} + A_{42}^{fg} \quad (13)$$

$$\begin{aligned} (\ln \mathcal{Z})_{62}^{fg} = & -540A_{11}^{fg^2} A_{20}^{fg^2} - 90A_{02}^{fg} A_{20}^{fg^3} + 90A_{20}^{fg^2} A_{22}^{fg} + 240A_{11}^{fg} A_{20}^{fg} A_{31}^{fg} - 20A_{31}^{fg^2} + 60A_{11}^{fg^2} A_{40}^{fg} \\ & + 30A_{02}^{fg} A_{20}^{fg} A_{40}^{fg} - 15(A_{22}^{fg} A_{40}^{fg} + A_{20}^{fg} A_{42}^{fg}) - 12A_{11}^{fg} A_{51}^{fg} - A_{02}^{fg} A_{60}^{fg} + A_{62}^{fg} \end{aligned} \quad (14)$$

$$\begin{aligned} (\ln \mathcal{Z})_{82}^{fg} = & 20160A_{11}^{fg^2} A_{20}^{fg^3} + 2520A_{02}^{fg} A_{20}^{fg^4} - 2520A_{20}^{fg^3} A_{22}^{fg} - 10080A_{11}^{fg} A_{20}^{fg^2} A_{31}^{fg} + 1120A_{20}^{fg} A_{31}^{fg^2} \\ & - 5040A_{11}^{fg^2} A_{20}^{fg} A_{40}^{fg} - 1260A_{02}^{fg} A_{20}^{fg^2} A_{40}^{fg} + 1120A_{11}^{fg} A_{31}^{fg} A_{40}^{fg} + 70A_{02}^{fg} A_{40}^{fg^2} - 70A_{40}^{fg} A_{42}^{fg} \\ & + 420(2A_{20}^{fg} A_{22}^{fg} A_{40}^{fg} + A_{20}^{fg^2} A_{42}^{fg}) + 672A_{11}^{fg} A_{20}^{fg} A_{51}^{fg} - 112A_{31}^{fg} A_{51}^{fg} + 112A_{11}^{fg^2} A_{60}^{fg} + 56A_{02}^{fg} A_{20}^{fg} A_{60}^{fg} \\ & - 28(A_{22}^{fg} A_{60}^{fg} + A_{20}^{fg} A_{62}^{fg}) - 16A_{11}^{fg} A_{71}^{fg} - A_{02}^{fg} A_{80}^{fg} + A_{82}^{fg} \end{aligned} , \quad (15)$$

with

$$A_2^f = \langle D_1^{f^2} \rangle + \langle D_2^f \rangle \quad (16)$$

$$A_4^f = \langle D_1^{f^4} \rangle + 6 \langle D_1^{f^2} D_2^f \rangle + 3 \langle D_2^{f^2} \rangle + 4 \langle D_1^f D_3^f \rangle + \langle D_4^f \rangle \quad (17)$$

$$\begin{aligned} A_6^f = & \langle D_1^{f^6} \rangle + 15 \langle D_1^{f^4} D_2^f \rangle + 45 \langle D_1^{f^2} D_2^{f^2} \rangle + 15 \langle D_2^{f^3} \rangle + 20 \langle D_1^{f^3} D_3^f \rangle \\ & + 60 \langle D_1^f D_2^f D_3^f \rangle + 10 \langle D_3^{f^2} \rangle + 15 \langle D_1^{f^2} D_4^f \rangle + 15 \langle D_2^f D_4^f \rangle + 6 \langle D_1^f D_5^f \rangle + \langle D_6^f \rangle \end{aligned} \quad (18)$$

$$\begin{aligned} A_8^f = & \langle D_1^{f^8} \rangle + 28 \langle D_1^{f^6} D_2^f \rangle + 210 \langle D_1^{f^4} D_2^{f^2} \rangle + 420 \langle D_1^{f^2} D_2^{f^3} \rangle + 105 \langle D_2^{f^4} \rangle + 56 \langle D_1^{f^5} D_3^f \rangle \\ & + 560 \langle D_1^{f^3} D_2^f D_3^f \rangle + 840 \langle D_1^f D_2^{f^2} D_3^f \rangle + 280 \langle D_1^{f^2} D_3^{f^2} \rangle + 280 \langle D_2^f D_3^{f^2} \rangle + 70 \langle D_1^{f^4} D_4^f \rangle \\ & + 420 \langle D_1^{f^2} D_2^f D_4^f \rangle + 210 \langle D_2^{f^2} D_4^f \rangle + 280 \langle D_1^f D_3^f D_4^f \rangle + 35 \langle D_4^{f^2} \rangle + 56 \langle D_1^{f^3} D_5^f \rangle \\ & + 168 \langle D_1^f D_2^f D_5^f \rangle + 56 \langle D_3^f D_5^f \rangle + 28 \langle D_1^{f^2} D_6^f \rangle + 28 \langle D_2^f D_6^f \rangle + 8 \langle D_1^f D_7^f \rangle + \langle D_8^f \rangle \end{aligned} \quad (19)$$

$$A_{11}^{fg} = \left\langle D_{01}^{fg} D_{10}^{fg} \right\rangle \quad (20)$$

$$A_{31}^{fg} = \left\langle D_{01}^{fg} D_{10}^{fg^3} \right\rangle + 3 \left\langle D_{01}^{fg} D_{10}^{fg} D_{20}^{fg} \right\rangle + \left\langle D_{01}^{fg} D_{30}^{fg} \right\rangle \quad (21)$$

$$\begin{aligned} A_{51}^{fg} &= \left\langle D_{01}^{fg} D_{10}^{fg^5} \right\rangle + 10 \left\langle D_{01}^{fg} D_{10}^{fg^3} D_{20}^{fg} \right\rangle + 15 \left\langle D_{01}^{fg} D_{10}^{fg} D_{20}^{fg^2} \right\rangle + 10 \left\langle D_{01}^{fg} D_{10}^{fg^2} D_{30}^{fg} \right\rangle + 10 \left\langle D_{01}^{fg} D_{20}^{fg} D_{30}^{fg} \right\rangle \\ &\quad + 5 \left\langle D_{01}^{fg} D_{10}^{fg} D_{40}^{fg} \right\rangle + \left\langle D_{01}^{fg} D_{50}^{fg} \right\rangle \end{aligned} \quad (22)$$

$$\begin{aligned} A_{71}^{fg} &= \left\langle D_{01}^{fg} D_{10}^{fg^7} \right\rangle + 21 \left\langle D_{01}^{fg} D_{10}^{fg^5} D_{20}^{fg} \right\rangle + 105 \left\langle D_{01}^{fg} D_{10}^{fg^3} D_{20}^{fg^2} \right\rangle + 105 \left\langle D_{01}^{fg} D_{10}^{fg} D_{20}^{fg^3} \right\rangle + 35 \left\langle D_{01}^{fg} D_{10}^{fg^4} D_{30}^{fg} \right\rangle \\ &\quad + 210 \left\langle D_{01}^{fg} D_{10}^{fg^2} D_{20}^{fg} D_{30}^{fg} \right\rangle + 105 \left\langle D_{01}^{fg} D_{20}^{fg^2} D_{30}^{fg} \right\rangle + 70 \left\langle D_{01}^{fg} D_{10}^{fg} D_{30}^{fg^2} \right\rangle + 35 \left\langle D_{01}^{fg} D_{10}^{fg^3} D_{40}^{fg} \right\rangle \\ &\quad + 105 \left\langle D_{01}^{fg} D_{10}^{fg} D_{20}^{fg} D_{40}^{fg} \right\rangle + 35 \left\langle D_{01}^{fg} D_{30}^{fg} D_{40}^{fg} \right\rangle + 21 \left\langle D_{01}^{fg} D_{10}^{fg^2} D_{50}^{fg} \right\rangle + 21 \left\langle D_{01}^{fg} D_{20}^{fg} D_{50}^{fg} \right\rangle \\ &\quad + 7 \left\langle D_{01}^{fg} D_{10}^{fg} D_{60}^{fg} \right\rangle + \left\langle D_{01}^{fg} D_{70}^{fg} \right\rangle \end{aligned} \quad (23)$$

$$A_{22}^{fg} = \left\langle D_{01}^{fg^2} D_{10}^{fg^2} \right\rangle + \left\langle D_{01}^{fg^2} D_{20}^{fg} \right\rangle + \left\langle D_{02}^{fg^2} D_{10}^{fg^2} \right\rangle + \left\langle D_{02}^{fg^2} D_{20}^{fg} \right\rangle \quad (24)$$

$$\begin{aligned} A_{42}^{fg} &= \left\langle D_{01}^{fg^2} D_{10}^{fg^4} \right\rangle + 6 \left\langle D_{01}^{fg^2} D_{10}^{fg^2} D_{20}^{fg} \right\rangle + 3 \left\langle D_{01}^{fg^2} D_{20}^{fg^2} \right\rangle + 4 \left\langle D_{01}^{fg^2} D_{10}^{fg} D_{30}^{fg} \right\rangle + \left\langle D_{01}^{fg^2} D_{40}^{fg} \right\rangle \\ &\quad + \left\langle D_{02}^{fg^2} D_{10}^{fg^4} \right\rangle + 6 \left\langle D_{02}^{fg^2} D_{10}^{fg^2} D_{20}^{fg} \right\rangle + 3 \left\langle D_{02}^{fg^2} D_{20}^{fg^2} \right\rangle + 4 \left\langle D_{02}^{fg^2} D_{10}^{fg} D_{30}^{fg} \right\rangle + \left\langle D_{02}^{fg^2} D_{40}^{fg} \right\rangle \end{aligned} \quad (25)$$

$$\begin{aligned} A_{62}^{fg} &= \left\langle (D_{01}^{fg^2} + D_{02}^{fg})(D_{10}^{fg^6} + 15D_{10}^{fg^4} D_{20}^{fg} + 15D_{20}^{fg^3} + 20D_{10}^{fg^3} D_{30}^{fg} + 10D_{30}^{fg^2} + 15D_{20}^{fg} D_{40}^{fg} \right. \\ &\quad \left. + 15D_{10}^{fg^2}(3D_{20}^{fg^2} + D_{40}^{fg}) + 6D_{10}^{fg}(10D_{20}^{fg} D_{30}^{fg} + D_{50}^{fg}) + D_{60}^{fg}) \right\rangle \end{aligned} \quad (26)$$

$$\begin{aligned} A_{82}^{fg} &= \left\langle (D_{01}^{fg^2} + D_{02}^{fg})(D_{10}^{fg^8} + 28D_{10}^{fg^6} D_{20}^{fg} + 105D_{20}^{fg^4} + 56D_{10}^{fg^5} D_{30}^{fg} + 210D_{20}^{fg^2} D_{40}^{fg} + 35D_{40}^{fg^2} \right. \\ &\quad \left. + 70D_{10}^{fg^4}(3D_{20}^{fg^2} + D_{40}^{fg}) + 56D_{30}^{fg} D_{50}^{fg} + 56D_{10}^{fg^3}(10D_{20}^{fg} D_{30}^{fg} + D_{50}^{fg}) + 28D_{20}^{fg}(10D_{30}^{fg^2} + D_{60}^{fg}) \right. \\ &\quad \left. + 28D_{10}^{fg^2}(15D_{20}^{fg^3} + 10D_{30}^{fg^2} + 15D_{20}^{fg} D_{40}^{fg} + D_{60}^{fg}) + 8D_{10}^{fg}(105D_{20}^{fg^2} D_{30}^{fg} + 35D_{30}^{fg} D_{40}^{fg} + 21D_{20}^{fg} D_{50}^{fg} \right. \\ &\quad \left. + D_{70}^{fg}) + D_{80}^{fg}) \right\rangle . \end{aligned} \quad (27)$$

Here we denote

$$D_i^f \equiv \frac{\partial^i}{\partial \mu_f^i} \left( \frac{1}{4} \ln \det M[\mu_u] + \frac{1}{4} \ln \det M[\mu_d] + \frac{1}{4} \ln \det M[\mu_s] \right) = \frac{1}{4} \frac{\partial^i}{\partial \mu_f^i} (\ln \det M[\mu_f]) \quad (28)$$

$$D_{ij}^{fg} \equiv \frac{\partial^i}{\partial \mu_f^i} \frac{\partial^j}{\partial \mu_g^j} \left( \frac{1}{4} \ln \det M[\mu_u] + \frac{1}{4} \ln \det M[\mu_d] + \frac{1}{4} \ln \det M[\mu_s] \right) = \begin{cases} D_i^f, & \text{if } i > 0, j = 0 \\ D_j^g, & \text{if } i = 0, j > 0 \\ 0, & \text{if } i > 0, j > 0 \end{cases} . \quad (29)$$

When restricting the light quark chemical potential to be equal ( $\mu_u = \mu_d \equiv \mu_q$ ) we define

$$D_i^f \equiv \frac{\partial^i}{\partial \mu_f^i} \left( \frac{2}{4} \ln \det M[\mu_q] + \frac{1}{4} \ln \det M[\mu_s] \right) \quad \text{and} \quad D_{ij}^{fg} \equiv \frac{\partial^i}{\partial \mu_f^i} \frac{\partial^j}{\partial \mu_g^j} \left( \frac{2}{4} \ln \det M[\mu_q] + \frac{1}{4} \ln \det M[\mu_s] \right) , \quad (30)$$

now with  $f, g \in \{q, s\}$ .